647510 **BSCalgCC5010** **Seat No :________**

B.Sc. Semester - 5 (Remedial) (*CBCS*) Examination
Feb/Mar. -2021 (OLD COURSE)
MATHEMATICAL ANALYSIS-1 AND ABSTRACT ALGEBRA-1(CORE)

Time: 1:30 Hours **Marks: 42**

Instructions:

1. Figures to the right indicate marks.
2. There are five questions in the question paper.
3. Answer any three of the following questions.

Q.1(A) Answer any one. (07)

1. State and prove the Housedorff's principle.
2. Prove that a subset E of a discrete metric space X is both open and closed in X.

Q.1(B) Answer any one. (04)

1. Let (X, d) be any metric space. Then show that $(X, \frac{d}{d+1})$ is also a metric space.
2. Prove that a finite subset of a metric space is always a closed set.

Q.1(C) Answer any one. (03)

1. Give an example of subsets A and B of metric space $\mathbb{R}$ such that $\overline{(A \cup B)} = \bar{A} \cup \bar{B}$.
2. Prove that $\frac{1}{4}$ is in the Cantor set.

Q.2(A) Answer any one. (07)

1. Let f be a bounded function defined on $[a, b]$. Let P and P^* be two partitions of $[a, b]$ such that $P \subset P^*$.
   Then, prove that $L(P, f) \leq L(P^*, f) \leq U(P^*, f) \leq U(P, f)$.
2. State and prove the necessary and sufficient conditions for a bounded function f on $[a, b]$ to be R-integrable.

Q.2(B) Answer any one. (04)

1. Prove that every constant function is R-integrable defined on any closed interval.
2. Let $f(x) = x^2$, $\forall x \in [0, 1]$.
   Evaluate $L(P, f)$ & $U(P, f)$ w.r.t. $P = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$.

Q.2(C) Answer any one. (03)

1. Define: (A) Upper & Lower Riemann sums. (B) Refinement of a partition
2. State Darboux's theorem.

Q.3(A) Answer any one. (07)

1. State and prove the fundamental theorem of integration.
2. Prove that identity element and inverse of an element in a group are always unique.

Q.3(B) Answer any one. (04)

1. Prove that $\frac{1}{2} < \int_0^{\frac{1}{2}} (1 - x^{2n})^{\frac{-1}{2}} \, dx < \frac{\pi}{6}$, *where* $n > 1$.
2. Let $(G, *)$ be any group and let $a, b \in G$.
   Prove that (A) $0(a^m) \leq 0(a)$, $\forall m \in \mathbb{Z}$.
   (B) $0(aba^{-1}) = 0(b)$.

Q.3(C) Answer any one. (03)

1. Prove that $\lim_{n\to\infty} (\frac{n^n}{n!})^{\frac{1}{n}} = e$.
2. Prove that a group G is commutative if $(ab)^2 = a^2b^2$, $\forall a, b \in G$.

Q.4(A) Answer any one. (07)

1. State and prove the Lagrange's theorem.
2. Prove that the composition of two disjoint cycles in S_n is commutative.

Q.4(B) Answer any one. (04)

1. State and prove the Fermat's theorem.
2. Let G be a finite group and let $a \in G$ be a fixed element of G.
   Then, prove that $f_a: G \to G$, $f_a(g) = ga$, $\forall g \in G$ is a permutation.

Q.4(C) Answer any one. (03)

1. If p is an odd prime, then show that
   $1^p + 2^p + \ldots.. + (p-1)^p \equiv 0(mod\ p)$.
2. If $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 5 & 2 & 6 & 4 \end{pmatrix} \in S_6$ then find $0(\sigma)$.

Q.5(A) Answer any one. (07)

1. Prove that a subgroup H of a group G is a normal subgroup
   $\Leftrightarrow aHa^{-1} \subset H, \forall a \in G$.
2. State and prove Cayley's theorem.

Q.5(B) Answer any one. (04)

1. Prove that the center Z of a group G is a normal subgroup of G.
2. Show that $(\mathbb{R}, +) \cong (\mathbb{R}_+, \cdot)$.

Q.5 (C) Answer any one. (03)

1. Let G be a group and let $H \leq G$.
   Then, prove that H is a normal subgroup of G if $(G:H) = 2$.
2. Let G be a group and let $a \in G$ be a fixed element of G.
   Then, show that the mapping $i_a: G \to G$, $i_a(g) = aga^{-1}$, $\forall\ g \in G$ is an isomorphism.

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